Journal of Statistical Physics, Vol. 72, Nos. 3/4, 1993

Phase Coexistence in Partially Symmetric q-State Models

Lahoussine Laanait,[1] Noureddine Masaif,[2] and Jean Ruiz[3]

Received October 30, 1992

We consider a lattice model whose spins may assume a finite number q of values. The interaction energy between two nearest-neighbor spins takes on the value $J_1 + J_2$ or J_2, depending on whether the two spins coincide or are different but coincide modulo q_1, and it is zero otherwise. This model is a generalization of the Ashkin–Teller model and exhibits the multilayer wetting phenomenon, that is, wetting by one or two or three interfacial layers, depending on the number of phases in coexistence. While we plan to consider interface properties in such a case, here we study the phase diagram of the model. We show that for large values of q_1 and q/q_1, it exhibits, according the value of J_2/J_1, either a unique first-order temperature-driven phase transition at some point β_t where q ordered phases coexist with the disordered one, or two transition temperatures $\beta_t^{(1)}$ and $\beta_t^{(2)}$, where q_1 partially ordered phases coexist with the ordered ones ($\beta_t^{(1)}$) or with the disordered one ($\beta_t^{(2)}$), or for a particular value of J_2/J_1 there is a unique transition temperature where all the previous phases coexist. Proofs are based on the Pirogov–Sinai theory: we perform a random cluster representation of the model (allowing us to consider noninteger values of q_1 and q/q_1) to which we adapt this theory.

KEY WORDS: Phase transitions; Pirogov–Sinai theory; random cluster models.

[1] Centre de Physique Théorique, CNRS, Luminy Case 907, 13288 Marseille Cedex 9, France. On leave from École Normale Supérieure de Rabat, B.P. 5118, Rabat, Morocco.

[2] Laboratoire de Physique Théorique, Département de Physique, Faculté des Sciences de Rabat, Morocco.

[3] Courant Institute of Mathematical Sciences, New York, New York, 10012, and Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903. On leave from Centre de Physique Théorique, CNRS, Marseille, France.

0022-4715/93/0800-0721\$07.00/0

1. INTRODUCTION

Dunlop *et al.*[1] introduced, for a microscopic study of the multilayer wetting phenomenon, a partially symmetric q-state model with Boltzmann weight

$$\exp\left[\beta \sum_{\langle i,j\rangle} J_1 \delta(x_i^1, x_j^1) + \beta \sum_{\langle i,j\rangle} J_2 \delta(x_i^1, x_j^1)\, \delta(x_i^2, x_j^2)\right] \tag{1}$$

Here β is the inverse temperature, J_1 and J_2 are nonnegative constants, the spin variables x_i^1 and x_i^2 defined on each site i of the $(d \geqslant 2)$-dimensional lattice $\mathbb{Z}^d$ belong, respectively, to the sets $\{1,..., q_1\}$ and $\{1,..., q_2\}$, the two sums are over nearest-neighbor pairs of a finite set Λ in $\mathbb{Z}^d$, and δ is the usual Kronecker symbol, $\delta(x, x') = 1$ if $x = x'$ and zero otherwise.

Introducing the probabilities $p_k = 1 - e^{-\beta J_k}$ for $k = 1$ and 2, we can write the corresponding probability distribution

$$\begin{aligned} d\mu(\mathbf{x}) = Z^{-1} \prod_{\langle i,j\rangle} &[(1-p_1)(1-p_2) + p_1(1-p_2)\, \delta(x_i^1, x_j^1) \\ &+ p_2 \delta(x_i^1, x_j^1)\, \delta(x_i^2, x_j^2)]\, d\mu_0(\mathbf{x}) \end{aligned} \tag{2}$$

where $d\mu_0$ is the counting measure over the configurations $\mathbf{x}$ of the spin variables in Λ.

This models exhibits three kinds of pure thermodynamic phases: q $(=q_1 q_2)$ ordered phases in which configurations $\{x_i^1\}$ and $\{x_i^2\}$ are both ordered, q_1 partially ordered phases in which only configurations $\{x_i^1\}$ are ordered while $\{x_i^2\}$ are disordered, and a disordered phase in which both configurations are disordered. The phase diagram shown in Fig. 1 was conjectured in ref. 1 on the basis of the analysis of restricted ensembles associated to the model. This expected phase diagram exhibits, in the plane $(\beta J_1, \beta J_2)$, three regions (where either only the q ordered phases coexist, or only the q_1 partially ordered phases coexist, or only the disordered phase is present) separated by three lines of phase coexistence (either between ordered and partially ordered phases, or between ordered and disordered phases, or between partially ordered and disordered phases) meeting in one point where all the phases coexist.

A proof of such results, which is the main purpose of this paper, might be obtained by using the extension of the Pirogov–Sinai theory[2] proposed by Bricmont *et al.*[3] as already noticed in ref. 1. However, here we will follow the idea of the approach recently applied by Laanait *et al.*[4] where the Pirogov–Sinai theory is adapted to the Fortuin–Kasteleyn representation[5] of the Potts model.

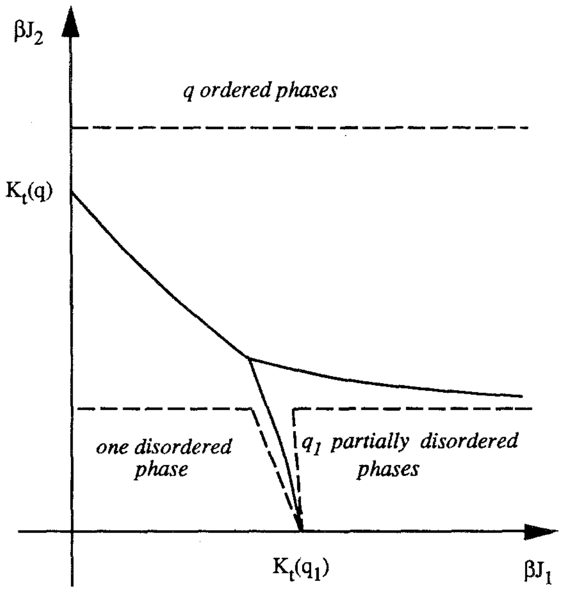

Fig. 1. The dashed lines bound the regions of the phase diagram already proven in ref. 1. $K_t(q)$ [resp. $K_t(q_1)$] is the transition point of the q (resp. q_1)-state Potts model.

In doing so, with the model under consideration, we first get a random cluster type model with two bond occupation variables n_{ij}^1 and n_{ij}^2, such that $n_{ij}^1 + n_{ij}^2 \leqslant 1$, with probability distribution

$$d\mu_{\mathrm{RC}}(\mathbf{n}) = Z_{\mathrm{RC}}^{-1} \prod_{\langle i,j\rangle \in G_f} (1-p_1)(1-p_2) \prod_{\langle i,j\rangle \in G_{pw}} p_1(1-p_2) \prod_{\langle i,j\rangle \in G_w} p_2 \times q_2^{N(n^2)} q_1^{N(n^1+n^2)}\, d\mu_0(\mathbf{n}) \qquad (3)$$

Here $\mathbf{n}$ denotes a configuration of the bond variables n_{ij}^1 and n_{ij}^2; G_f is the set of bonds ij, to be called *empty*, having $n_{ij}^1 = n_{ij}^2 = 0$; G_{pw} is the set of bonds ij, to be called *partially wired*, having $n_{ij}^1 = 1 - n_{ij}^2 = 1$; G_w is the set of bonds ij, to be called *wired*, having $n_{ij}^2 = 1 - n_{ij}^1 = 1$; $N(n^2)$ [respectively $N(n^1 + n^2)$] denotes the number of connected components, including isolated sites, in the graph whose edges are the bonds having $n_{ij}^2 = 1$ (respectively $n_{ij}^1 + n_{ij}^2 = 1$); $d\mu_0$ is the counting measure.

For any configuration $\mathbf{n} = (G_f, G_{pw}, G_w)$, we shall use $S_p(\mathbf{n})$ to denote the set of sites i where all bonds with endpoint i belong to G_p for $p \in \{f, pw, w\}$. We shall use $S(n^k)$, for $k = 1, 2$, to denote the set of sites that are endpoints of bonds having $n_{ij}^k = 1$, and use $C(n^2)$ [respectively $C(n^1 + n^2)$] to denote the number of connected components in the graph whose edges are the bonds having $n_{ij}^2 = 1$ (respectively $n_{ij}^1 + n_{ij}^2 = 1$), and use $|E|$ to denote the number of elements of the set E. With these notations the Boltzmann weight of a configuration $\mathbf{n}$ can be written (cf. Section 2)

$$e^{-H(\mathbf{n})} = q^{|S_f(\mathbf{n})|} q_2^{[|S(n^1)| - |S(n^1) \cap S(n^2)|]} (e^{\beta J_1} - 1)^{\Sigma_{\langle ij \rangle} n_{ii}^1} \\ \times [e^{\beta J_1}(e^{\beta J_2} - 1)]^{\Sigma_{\langle ij \rangle} n_{ij}^2} q_2^{C(n^2)} q_1^{C(n^1 + n^2)} \tag{4}$$

up to the constant term $\prod_{\langle ij \rangle} e^{-\beta(J_1 + J_2)}$ that we introduced in (2) and (3) to make more transparent the probabilistic interpretation of the models.

It turns out that for $e^{\beta J_1}(e^{\beta J_2} - 1) = (e^{\beta J_1} - 1)\, q_2^{1/d} = q^{1/d}$ the Hamiltonian H has three ground states:

1. The configuration on $\mathbb{Z}^d$ with only empty bonds, $\mathbf{n}_f$, with energy per site[4]

$$e_f = -\log q \tag{5}$$

2. The configuration on $\mathbb{Z}^d$ with only partially wired bonds, $\mathbf{n}_{pw}$, with energy per site

$$e_{pw} = -\log q_2 - d \log(e^{\beta J_1} - 1) \tag{6}$$

3. The configuration on $\mathbb{Z}^d$ with only wired bonds, $\mathbf{n}_w$, with energy per site

$$e_w = -d\beta J_1 - d \log(e^{\beta J_2} - 1) \tag{7}$$

The ground states $\mathbf{n}_f$ and $\mathbf{n}_{pw}$ represent, respectively, the free energy of the disordered and partially disordered states of the original model. Together with $\mathbf{n}_w$ they are actually the only ground states of the Hamiltonian H for nonnegative values of J_1 and J_2; the phase diagram of ground states inferred from (5)–(7) is shown in Fig. 2.

Indeed, it is easy to see that, for any configuration $\mathbf{n}$ on $\mathbb{Z}^d$ that differs from one of the configurations $\mathbf{n}_p$, $p \in \{f, pw, w\}$, only on a finite set of bonds, $\mathbf{n} = \mathbf{n}_p$ (a.s.), the *relative energy* $H(\mathbf{n} \mid \mathbf{n}_p) = H(\mathbf{n}) - H(\mathbf{n}_p)$ satisfies $H(\mathbf{n} \mid \mathbf{n}_p) \geqslant 0$ whenever $e_p = \min\{e_f, e_{pw}, e_w\}$; moreover, this relative energy satisfies the *Peierls condition* (cf. Section 3)

$$H(\mathbf{n} \mid \mathbf{n}_p) \geqslant \frac{|\partial \mathbf{n}|}{2d} \log(\min\{q_1, q_2\}) \tag{8}$$

[4] The energy per site of a translation configuration $\mathbf{n}$ is defined as the thermodynamic limit of the ratio between the energy $H(\mathbf{n})$ and the number of sites in Λ.

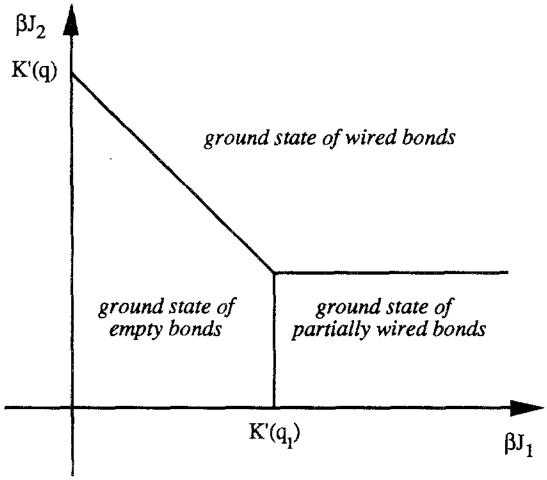

Fig. 2. Diagram of ground states of the Hamiltonian (4); $K'(q) = \log(q^{1/d} + 1)$ and $K'(q_1) = \log(q_1^{1/d} + 1)$.

where $\partial \mathbf{n}$, to be called the *boundary of the configuration* $\mathbf{n}$, is defined as $\partial \mathbf{n} = \mathbb{Z}^d \backslash \bigcup_{p \in \{f, pw, w\}} S_p(\mathbf{n})$.

Hence, with the help of the Pirogov–Sinai theory, a sketch of which we present for the model under consideration in Section 3, with a slightly different definition of contours more suitable for our purpose, we get that the phase diagram of the random cluster model (3) mimics, for large values of q_1 and q_2, the phase diagram of ground states. As a result we get also further information on both models, which we discuss in Section 4.

To close this introduction, we notice that by analogy with Edwards and Sokal, who provided in ref. 6 a simple explanation of the Swendsen–Wang algorithm,[7] we can also consider a joint model having two Potts spin variables x_i^1 and x_i^2 at sites, and two occupation variables n_{ij}^1 and n_{ij}^2 on bonds, such that $n_{ij}^1 + n_{ij}^2 \leqslant 1$, with probability distribution

$$
\begin{aligned}
d\mu_{\text{joint}}(\mathbf{x}, \mathbf{n}) = Z_{\text{joint}}^{-1} \prod_{\langle i,j \rangle} & [(1-p_1)(1-p_2)\, \delta(n_{ij}^1, 0)\, \delta(n_{ij}^2, 0) \\
& + p_1(1-p_2)\, \delta(n_{ij}^1, 1)\, \delta(n_{ij}^2, 0)\, \delta(x_i^1, x_j^1) \\
& + p_2 \delta(n_{ij}^1, 0)\, \delta(n_{ij}^2, 1)\, \delta(x_i^1, x_j^1)\, \delta(x_i^2, x_j^2)] \\
& \times d\mu_0(\mathbf{x})\, d\mu_0(\mathbf{n})
\end{aligned} \tag{9}
$$

As we shall see more precisely in Section 2, the summation over either the $\mathbf{x}$ or $\mathbf{n}$ variables gives (1) $Z = Z_{\text{RC}} = Z_{\text{joint}}$; (2) the marginal distribution of the Potts sites variables $\mathbf{x}$ is precisely the model $d\mu$; and (3) the marginal distribution of the bond occupation variables $\mathbf{n}$ is precisely the random cluster model $d\mu_{\mathrm{RC}}$.

It would be interesting to use the efficient Swendsen–Wang algorithm in the case $q_1 = q_2 = 2$ describing the Ashkin–Teller model, where several open problems about the phase diagram[8] and critical exponents on transition lines[9] occur. Let us mention that, for this model, a proof of the existence of an intermediate region for some values of J_2/J_1 is given in ref. 10.

We finally mention that the equivalence stated above between the different representations of the model is neither restricted to nearest neighbors nor to equal coupling constants. This equivalence is proved in Section 2, where we introduce also partition functions with boundary conditions associated to the Hamiltonian (4). Section 3 contains our main results and Section 4 our concluding remarks. The proof of the Peierls condition is given in the Appendix.

2. THE RANDOM CLUSTER EXPANSION

In this section we prove the equality between partition functions stated in the introduction, in particular, that the sum of the Boltzmann factor (1) over the configurations $\mathbf{x}$ equals the sum of the Boltzmann factor (4) over the configurations $\mathbf{n}$.

To prove this equivalence, we shall apply the formula $e^{\beta J \delta} = 1 + (e^{\beta J} - 1)\delta$ to each bond successively. More precisely, we use the formula

$$\begin{aligned}
&\exp[\beta J_1 \delta(x_i^1, x_j^1) + \beta J_2 \delta(x_i^1, x_j^1)\, \delta(x_i^2, x_j^2)] \\
&\quad = \{1 + [\exp(\beta J_2) - 1]\, \delta(x_i^1, x_j^1)\, \delta(x_i^2, x_j^2)\} \exp[\beta J_1 \delta(x_i^1, x_j^1)] \\
&\quad = \exp[\beta J_1 \delta(x_i^1, x_j^1)] \\
&\qquad + [\exp(\beta J_1)][\exp(\beta J_2) - 1]\, \delta(x_i^1, x_j^1)\, \delta(x_i^2, x_j^2)
\end{aligned}$$

for each bond, so that the sum of the Boltzmann factor (1), say $\tilde{Z}$, can be written

$$\begin{aligned}
\tilde{Z} &= \{\exp[\beta(J_1 + J_2)\, |L(\Lambda)|]\} Z \\
&= \sum_{\mathbf{x}} \sum_{G_w \subset L(\Lambda)} \{[\exp(\beta J_1)][\exp(\beta J_2) - 1]\}^{|G_w|} \prod_{\langle ij \rangle \in G_w} \delta(x_i^1, x_j^1)\, \delta(x_i^2, x_j^2) \\
&\quad \times \prod_{\langle ij \rangle \in L(\Lambda) \backslash G_w} \exp[\beta J_1 \delta(x_i^1, x_j^1)] \qquad (10)
\end{aligned}$$

where $L(\Lambda)$ is the set of bonds with two endpoints in Λ. By applying again the above-mentioned formula for $e^{\beta J \delta}$ to each bond in $L(\Lambda)\backslash G_w$, we get

$$\tilde{Z} = \sum_{\mathbf{x}} \sum_{G_w \subset L(\Lambda)} \sum_{G_{pw} \subset L(\Lambda)\backslash G_w} [e^{\beta J_1}(e^{\beta J_2} - 1)]^{|G_w|} (e^{\beta J_1} - 1)^{|G_{pw}|}$$
$$\times \prod_{\langle ij \rangle \in G_w} \delta(x_i^1, x_j^1)\, \delta(x_i^2, x_j^2) \prod_{\langle ij \rangle \in G_{pw}} \delta(x_i^1, x_j^1) \tag{11}$$

Notice that formula (11) actually gives the partition function of the joint model (up to the constant term $e^{\beta(J_1 + J_2)|L(\Lambda)|}$).

To get the equivalence with the random cluster model, we invert the summations in (11) and use

$$\sum_{\mathbf{x}} \prod_{\langle ij \rangle \in G_w} \delta(x_i^1, x_j^1)\, \delta(x_i^2, x_j^2) \prod_{\langle ij \rangle \in G_{pw}} \delta(x_i^1, x_j^1)$$
$$= \sum_{\mathbf{x}} \prod_{\langle ij \rangle \in G_w} \delta(x_i^2, x_j^2) \prod_{\langle ij \rangle \in G_w \cup G_{pw}} \delta(x_i^1, x_j^1) = q_2^{N(n^2)} q_1^{N(n^1 + n^2)}$$
$$= q_2^{C(n^2) + |\Lambda| - |S(n^2)|} q_1^{C(n^1 + n^2) + |\Lambda| - |S(n^1) \cup S(n^2)|} \tag{12}$$

Indeed, from (11) and (12), we infer that $\tilde{Z}$ is the sum of the Boltzmann factor (4) by taking into account that $|G_{pw}| = \sum_{\langle ij \rangle} n_{ij}^1$ and $|G_w| = \sum_{\langle ij \rangle} n_{ij}^2$, together with the equalities

$$|\Lambda| - |S(n^1) \cup S(n^2)| = |S_f(\mathbf{n})|$$
$$|S(n^1) \cup S(n^2)| - |S(n^2)| = |S(n^1)| - |S(n^1) \cap S(n^2)|$$

Up to now, we have considered only free boundary conditions. To formulate our results we shall introduce other boundary conditions that force the system to be in a stable phase, and for which the equivalence can also be shown.

For any set Λ in $\mathbb{Z}^d$, we define the *envelope* of Λ, $E(\Lambda)$, as the set of bonds having one or two endpoints in Λ, and the *boundary* of Λ, $\partial\Lambda$, as the set of sites in Λ having a nearest neighbor in $\mathbb{Z}^d \backslash \Lambda$. We shall use $\Omega(\Lambda)$ to denote the set of configurations $\mathbf{n}$ on $E(\Lambda)$ and for $p \in \{f, pw, w\}$, we shall use $\Omega^p(\Lambda)$ to denote the set of configurations $\mathbf{n} \in \Omega(\mathbb{Z}^d) \equiv \Omega$ such that all the sites of $\partial\Lambda$ and $\mathbb{Z}^d \backslash \Lambda$ belong to $S_p(\mathbf{n})$.

We define the energy per site of a configuration $\mathbf{n} \in \Omega$ by

$$e_i(\mathbf{n}) = -\left[\prod_{\langle ij \rangle \in E(i)} (1 - n_{ij}^1)(1 - n_{ij}^2)\right] \log q - [\chi(i \in S(n^1), i \notin S(n^2)] \log q_2$$
$$- \frac{\xi_1}{2}\, \delta n^1(i) - \frac{\xi_2}{2}\, \delta n^2(i) \tag{13}$$

where χ is the characteristic function, δ is the codifferential operator $\delta n^k(i) = \sum_{\langle ij \rangle \in E(i)} n_{ij}^k$ for $k = 1, 2$, and

$$e^{\xi_1} = e^{\beta J_1} - 1 \tag{14}$$

$$e^{\xi_2} = e^{\beta J_1}(e^{\beta J_2} - 1) \tag{15}$$

We define the Hamiltonian in Λ of a configuration $\mathbf{n} = \mathbf{n}_p$ (a.s.) by

$$H_\Lambda(\mathbf{n}) = \sum_{i \in \Lambda} e_i(\mathbf{n}) + \tilde{H}(\mathbf{n})$$

$$\tilde{H}(\mathbf{n}) \equiv -C(n^2) \log q_2 - C(n^1 + n^2) \log q_1 + C_p$$

where $C_f = 0$, $C_{pw} = \log q_1$, and $C_w = \log q$. Let us observe that for any configuration $\mathbf{n} \in \Omega^p(\Lambda)$ the Hamiltonian $H_\Lambda(\mathbf{n})$ coincides with that given by (4) up to a boundary term and to the term C_p. The boundary term is the usual difference between the "physical" and "diluted" partition functions in the Pirogov–Sinai theory. We shall introduce only the second one, which is more convinient for our purpose. The term $-C_p$ represents the contribution to connected components of the configuration $\mathbf{n}_p$. Subtracting this term will allow us to write expansions of the Hamiltonian [see (22)] and partition functions in a form where the different components of the boundary of a configuration (contours) do not interact. Let us also mention that for any configuration $\mathbf{n} \in \Omega^p(\Lambda)$,

$$H(\mathbf{n} \mid \mathbf{n}_p) = H_\Lambda(\mathbf{n}) - H_\Lambda(\mathbf{n}_p) = H_\Lambda(\mathbf{n}) - e_p\, |\Lambda| \tag{16}$$

since $e_i(\mathbf{n}_p) = e_p$.

For each $p \in \{f, pw, w\}$, we define the *diluted partition function*

$$Z_p^{\text{dil}}(\Lambda) = \sum_{\mathbf{n} \in \Omega^p(\Lambda)} e^{-H_\Lambda(\mathbf{n})} \tag{17}$$

and introduce the thermodynamic limit

$$s(H) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z_p^{\text{dil}}(\Lambda) \tag{18}$$

which is independant of the boundary condition p.

3. MAIN RESULTS

We consider a configuration $\mathbf{n} = \mathbf{n}_p$ (a.s.), and recall that the boundary $\partial\mathbf{n}$ of $\mathbf{n}$ is defined as the complement of the set of sites i where all the bonds with endpoint i are either empty or partially wired or wired.

A couple $\gamma = \{\Gamma, \mathbf{n}(\Gamma)\}$, where $\Gamma \equiv \operatorname{Supp} \gamma$ is a maximal connected subset[5] (component) of $\partial\mathbf{n}$ and $\mathbf{n}(\Gamma)$ a configuration in $\Omega(\Gamma)$ [i.e., a configuration on the envelope $E(\Gamma)$ of Γ], is called a *contour* of the configuration $\mathbf{n}$. A couple $\gamma = \{\Gamma, \mathbf{n}(\Gamma)\}$, where Γ is a connected subset of $\mathbb{Z}^d$ and $\mathbf{n}(\Gamma)$ a configuration in $\Omega(\Gamma)$, is called a contour if there exists a configuration $\mathbf{n}$ such that γ is a contour of $\mathbf{n}$.

Whenever γ is a contour, we denote by $\operatorname{Ext} \gamma$ the unique infinite component of $\mathbb{Z}^d \backslash \operatorname{Supp} \gamma$ and $V(\gamma) = \mathbb{Z}^d \backslash \operatorname{Ext} \gamma$ and $\operatorname{Int} \gamma = V(\gamma) \backslash \operatorname{Supp} \gamma$. Consider the configuration $\mathbf{n}$ having γ as unique contour, to be denoted $\mathbf{n}_\gamma$; we use $\operatorname{Int}_m \gamma$, for any $m \in \{f, pw, w\}$, to denote the subset of sites of $\operatorname{Int} \gamma$ which belong to $S_m(\mathbf{n})$. When this configuration will coincide with $\mathbf{n}_p$ on the envelope of $\operatorname{Ext} \gamma$, we shall specify this with a subscript p.

Two contours γ_i and γ_j with nonconnected supports are called mutually compatible contours. They are mutually compatible external contours if $V(\gamma_i) \subset \operatorname{Ext} \gamma_j$ and $V(\gamma_j) \subset \operatorname{Ext} \gamma_i$. We shall use $\Omega(\gamma^p)$ to denote the set of configurations having γ^p as unique external contour, and for a family $\theta^p = \{\gamma_1^p, ..., \gamma_n^p\}$ of external contours, we shall use the notation $\operatorname{Ext}_\Lambda \theta^p = \Lambda \backslash \bigcup_{\gamma^p \in \theta^p} V(\gamma^p)$. To simplify formulas we shall let the symbol γ or γ^p (respectively θ^p) denote a contour (respectively, a family of external contour) as well as its support; in particular we shall use H_γ instead of $H_{\operatorname{Supp} \gamma}$.

We introduce the *crystal partition function*

$$Z^{\mathrm{cr}}(\gamma^p) = \sum_{\mathbf{n} \in \Omega(\gamma^p)} \exp[-H_{V(\gamma^p)}(\mathbf{n})] \tag{19}$$

for which, together with the diluted partition function (17), the following set of recurrence equations holds:

$$Z_p^{\mathrm{dil}}(\Lambda) = \sum_{\substack{\theta^p : \theta^p \subset \Lambda \\ \theta^p \cap \partial\Lambda = \varnothing}} \exp(-e_p \, |\operatorname{Ext}_\Lambda \theta^p|) \prod_{\gamma^p \in \theta^p} Z^{\mathrm{cr}}(\gamma^p) \tag{20}$$

$$Z^{\mathrm{cr}}(\gamma^p) = \exp[-H_{\gamma^p}(\mathbf{n}_{\gamma^p})] \prod_m Z_m^{\mathrm{dil}}(\operatorname{Int}_m \gamma^p) \tag{21}$$

where the sum runs over families of external contours with support included in Λ and nonintersecting $\partial\Lambda$. This is because for every configuration $\mathbf{n} = \mathbf{n}_p$ (a.s.), specified by a family of contour $\{\gamma\}$, one has

$$H_\Lambda(\mathbf{n}) = \sum_\gamma H_\gamma(\mathbf{n}_\gamma) + \sum_m e_m \, |S_m(\mathbf{n})| \tag{22}$$

[5] A set of sites Γ is connected if for every two elements $i, j \in \Gamma$, there is a sequence $i = i_1, i_2, ..., i_n = j$ such that i_k and i_{k+1} are nearest neighbors for $k = 1, ..., n-1$.

for every $\Lambda \supset \{\gamma\}$, since $\tilde{H}(\mathbf{n}) = \sum_\gamma \tilde{H}(\mathbf{n}_\gamma)$. The recurrence equations (20) and (21) are equivalent to Lemma (2.3) in ref. 2, where the partition functions are defined with relative Hamiltonian.

Lemma. Let $q_0 = \min\{q_1, q_2\}$ and $e_0 = \min\{e_f, e_{pw}, e_w\}$; then

$$H_\gamma(\mathbf{n}_\gamma) - e_0 |\gamma| \geqslant \frac{|\gamma|}{2d} \log q_0 \tag{23}$$

We postpone the proof to the Appendix. Let us mention that, whenever $e_p = e_0$, the Lemma implies

$$\rho(\gamma^p) \equiv \exp[-H_{\gamma^p}(\mathbf{n}_{\gamma^p}) + e_p |\gamma^p|] \leqslant q_0^{-(1/2d)|\gamma^p|} \tag{24}$$

and also the Peierls condition (8) by taking into account (16) and (22).

To state our result, we introduce for each p the partition function of a *contour model* with a parameter b_p and contour weight $\phi_p^{b_p}(\gamma^p)$,

$$Z(\Lambda \mid \phi_p^{b_p}; b_p) = \sum_{\substack{\theta^p : \theta^p \subset \Lambda \\ \theta^p \cap \partial\Lambda = \varnothing}} e^{b_p V(\theta^p)} \prod_{\gamma^p \in \theta^p} \phi_p^{b_p}(\gamma^p)\, Z(\operatorname{Int} \gamma^p \mid \phi_p^{b_p}) \tag{25}$$

Here

$$Z(\Lambda \mid \phi_p^{b_p}) = \sum_{\substack{\partial : \partial \subset \Lambda \\ \partial \cap \partial\Lambda = \varnothing}} \prod_{\gamma^p \subset \partial} \phi_p^{b_p}(\gamma^p) \tag{26}$$

where the sum is over families of compatibles contours. The functional $\phi_p^{b_p}$ is called a τ-functional if for some number $\tau > 0$ and every γ it satisfies the estimates $|\phi_p^{b_p}(\gamma^p)| \leqslant e^{-\tau |\gamma^p|}$; this ensures in particular the existence of the thermodynamic limit

$$s(\phi_p^{b_p}) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z(\Lambda \mid \phi_p^{b_p})$$

Theorem. Whenever q_0 is large enough, then for every nonnegative value of J_1 and J_2 and every $p \in \{f, pw, w\}$, there exist nonnegative parameters b_p and associated contour functionals $\phi_p^{b_p}$ such that

$$b_p - e_p + s(\phi_p^{b_p}) = s(H) \tag{27}$$

$$e^{-e_p |\Lambda|} Z(\Lambda \mid \phi; b) = Z_p^{\text{dil}}(\Lambda) \tag{28}$$

There exist three regions in the plane $(\beta J_1, \beta J_2)$ where $b_p = 0$, $b_{p' \neq p} > 0$, separated by three trajectories of phase coexistence determined by the equations

$$-e_f + s(\phi_f) = -e_{pw} + s(\phi_{pw}) = s(H) \tag{29}$$

$$-e_{pw} + s(\phi_{pw}) = -e_w + s(\phi_w) = s(H) \tag{30}$$

$$-e_w + s(\phi_w) = -e_f + s(\phi_f) = s(H) \tag{31}$$

meeting at a unique point where $b_f = b_{pw} = b_w = 0$.

The contour functionals $\phi_p^{b_p}$ defined inductively by

$$\phi_p^{b_p}(\gamma^p)\, Z(\mathrm{Int}\, \gamma^p \mid \phi_p^{b_p}) = e^{-(b_p - e_p)\,|V(\gamma^p)|} Z^{\mathrm{cr}}(\gamma^p) \tag{32}$$

satisfy the estimates

$$\phi_p^{b_p}(\gamma^p) \leqslant q_0^{-(1/2d)\,|\gamma^p|} \exp(9 q_0^{-1/2d}\, |\gamma^p|) \tag{33}$$

Proof. We refer the reader to ref. 11 for a proof given for the general class of models of the Pirogov–Sinai theory satisfying the Peierls condition. We also refer to ref. 12, where a proof is given for the model under consideration. We only mention that the inductive expression (32) of contour functionals immediately yields relation (28) of the theorem, taking into account the inductive expressions (20) and (21) and the definition (25). The statement (27) follows also from (32) provided the $\phi_p^{b_p}$ are indeed τ-functionals. We also mention that, according to ref. 11 and (23), the bound (33) follows from the inequality

$$\phi_p^{b_p}(\gamma^p) \leqslant q_0^{-(1/2d)\,|\gamma^p|} \exp(3 e^{-\tau}\, |\gamma^p|) \tag{34}$$

where τ must satisfy the inequality

$$q_0^{-(1/2d)\,|\gamma^p|} \exp(3 e^{-\tau}\, |\gamma^p|) \leqslant \exp(-\tau\, |\gamma^p|) \tag{35}$$

This is done by taking $e^{-\tau} = 3 q_0^{-(1/2d)}$ provided $q_0 > 3^{2d}$. ∎

4. CONCLUDING REMARKS

The Theorem above shows that the random cluster model (3) may be described equivalently as a system of noninteracting compatible contours. If q_0 is large enough, the associated activities are small and decay exponentially in the contour length. This allows a good control of the system *at any temperature.*

We introduce the state $\langle \cdot \rangle^\alpha$ for $\alpha \in \{w, pw, f\}$. When $b_f = 0$, we have

$\langle n_{ij}^1\rangle^f \leqslant O(q_0^{1/d})$ and $\langle n_{ij}^2\rangle^f \leqslant O(q_0^{1/d})$ for any bond ij. This is because $n_{ij}^1 = 1$ (resp. $n_{ij}^2 = 1$) only if there is a contour surrounding or containing i or j, ϕ_f is an upper bound on the probability of contour, and the shortest contour has length $|\gamma| = 2$. Analogously, one has the same upper bound for $\langle 1 - n_{ij}^1\rangle^{pw}$ and $\langle n_{ij}^2\rangle^{pw}$ when $b_{pw} = 0$, and for $\langle 1 - n_{ij}^1\rangle^w$ and $\langle 1 - n_{ij}^2\rangle^w$ when $b_w = 0$. From this we get that the coexistence line is first order with a jump $\Delta E > 0$ of the internal energy $E = (1/\beta)\, \partial s(H)/\partial\beta$. Notice that in the particular cases $J_1 = 0$ and $J_2 = 0$, we get, respectively, the q- and q_1-state Potts models, to which our results apply.

The phase diagram can be obtained also, following ref. 13, without introducing parametric contour models. Let ξ_1^0 and ξ_2^0 be the values of the parameters ξ_1 and ξ_2 such that $e_f = e_{pw} = e_w$. Then the differences $\mu_1 \equiv d(\xi_1 - \xi_1^0) = e_{pw} - e_f$ and $\mu_2 \equiv d(\xi_2 - \xi_2^0) = e_w - e_f$ may be considered as generalized external fields. The required degeneracy-breaking condition for the matrix

$$\left[\frac{\partial}{\partial \mu_i}(e_p - e_f)\right]_{p = pw, w;\, i = 1, 2} \tag{36}$$

to be nonsingular, as well as the condition $|\partial e_p/\partial \mu_i| \leqslant 1$,[14] are obviously satisfied. Then (using the notations of ref. 14), we introduce the functionals $K(\gamma^p) = \rho(\gamma^p)\, f_m(\gamma^p)$, where

$$f_m(\gamma^p) = [Z_p^{\mathrm{dil}}(\mathrm{Int}_m\, \gamma^p)]^{-1}\, Z_m^{\mathrm{dil}}(\mathrm{Int}_m\, \gamma^p)$$

so that

$$Z_p^{\mathrm{dil}}(\Lambda) = e^{-e_p |\Lambda|} \sum_{\substack{\partial \subset \Lambda \\ \partial \cap \partial\Lambda = \varnothing}} \prod_{\gamma^p \in \partial} K(\gamma^p) \tag{37}$$

We introduce also the truncated partition function $Z_q'(\Lambda)$ as the right-hand side of (37) with a sum over stable contours, i.e., satisfying $f_m(\gamma^p) \leqslant \exp(4\, |\partial\, \mathrm{Int}_m\, \gamma^p|)$, and the corresponding free energy h_p. According to refs. 13 and 14, for q_0 large enough, the Peierls condition (24) ensures that for stable boundary conditions p, i.e., satisfying $a_p = h_p - \min_m h_m = 0$, then the contours γ^p are stable and thus Z_p^{dil} agree with Z_p'. As a result, we obtain the same description as in the Theorem with the role of parameters b_p replaced by a_p.

The completeness of the phase diagram or its differentiability properties allow us to derive some properties of the model (2) by taking into account the analysis of translation-invariant Gibbs states of refs. 15 and 16. Further differentiability properties of the phase diagram follow from ref. 17.

Finally, we introduce convenient mixed boundary conditions to define the surface tensions between two wired phases, $\sigma^{w,w'}$, and between two partially wired phases, $\sigma^{pw,pw'}$, and a surface tension between a wired (resp. partially wired) phase and the empty phase, $\sigma^{w,f}$ (resp. $\sigma^{pw,f}$), and also surface tensions between wired and partially wired phases, $\sigma^{w,pw}$. This is done by restricting the allowed configuration of bonds (with $n^1_{ij}=1$, $n^2_{ij}=0$ or $n^1_{ij}=0$, $n^2_{ij}=1$) to those which do not connect a site of the top half boundary to a site of the bottom half boundary. We use these nonconnectedness conditions and the Theorem to prove that the surface tensions between coexisting stable phases are strictly positive.[12] Moreover, our goal will be to prove that these surface tensions satisfy the (generalized) *Antonov rule*

$$\begin{aligned}
\sigma^{w,w'} &= \sigma^{w,f} + \sigma^{f,w'} \\
\sigma^{w,w'} &= \sigma^{w,pw} + \sigma^{pw,w'} + \sigma^{pw',w'} \qquad (38)\\
\sigma^{w,w'} &= \sigma^{w,pw} + \sigma^{pw,f} + \sigma^{f,pw'} + \sigma^{pw',w'}
\end{aligned}$$

when the considered phases coexist. A first important step has been given in ref. 1 (see also ref. 18), where it is proven by correlation inequalities that the left-hand sides are greater than the right-hand sides for the three equations in (38). The converse inequalities need a detailed analysis of interfaces following ideas of ref. 19. The analysis given in this article provides a step in this direction.

APPENDIX. PROOF OF THE LEMMA

Consider a site $i \in \gamma$, and first assume that there is no wired bond in $E(i)$ for the configuration $\mathbf{n}_{\gamma p}$. Then $E(i)$ contains at least one partially wired bond. Starting from (13) and using $e_0 \leqslant -d\xi_1 - \log q_2$ [cf. (6) and (14)] and $e_0 \leqslant -\log q_1 - \log q_2$ [cf. (5)], we get

$$\begin{aligned}
e_i(\mathbf{n}) - e_0 &= -\frac{\xi_1}{2}\,\delta n^1(i) - \log q_2 - e_0 \\
&\geqslant \frac{e_0 + \log q_2}{2d}\,\delta n^1(i) - \frac{e_0 + \log q_2}{2d}\,2d \\
&\geqslant \frac{2d - \delta n^1(i)}{2d}\log q_1
\end{aligned}$$

We then use that the number of connected components $C(n^1+n^2)$ satisfies the bound (see Appendix B in ref. 20 for details)

$$C(n^1+n^2) \leqslant \sum_{i:1 \leqslant \delta n^1(i) + \delta n^2(i) \leqslant d} \frac{1}{2^{|\delta n^1(i) + \delta n^2(i)|}} \qquad \text{(A1)}$$

When $1 \leqslant \delta n^1(i) \leqslant 2d-1$ and $\delta n^2(i)=0$ we have

$$\frac{2d-\delta n^1(i)}{2d}\log q_1 - \frac{\chi(1 \leqslant \delta n^1(i) \leqslant d)}{2^{|\delta n^1(i)|}}\log q_1 \geqslant \frac{1}{2d}\log q_1 \tag{A2}$$

so that each site $i \in \gamma$ satisfying $\delta n^2(i)=0$ gives at least a contribution $(1/2d)\log q_1$ to the left-hand side of (23).

Now we assume that at least a bond in $E(i)$ is wired. Starting from (13) and using $e_0 \leqslant -d\xi_1 - \log q_2$ and $e_0 \leqslant -\xi_2$ [cf. (7) and (15)] and $e_0 \leqslant -\log q_1 - \log q_2$ we get

$$\begin{aligned} e_i(\mathbf{n}) - e_0 &= -\frac{\xi_1}{2}\delta n^1(i) - \frac{\xi_2}{2}\delta n^2(i) - e_0 \\ &\geqslant \frac{e_0}{2d}[\delta n^1(i) + \delta n^2(i)] + \frac{\log q_2}{2d}\delta n^1(i) - e_0 \\ &\geqslant \frac{2d - \delta n^1(i) - \delta n^2(i)}{2d}\log q_1 + \frac{2d - \delta n^2(i)}{2d}\log q_2 \end{aligned}$$

We then use that the number of connected components $C(n^2)$ satisfies the bound

$$C(n^2) \leqslant \sum_{i:1 \leqslant \delta n^2(i) \leqslant d} \frac{1}{2^{|\delta n^2(i)|}} \tag{A3}$$

When $1 \leqslant \delta n^2(i) \leqslant 2d-1$ we infer

$$\begin{aligned} &\frac{2d - \delta n^1(i) - \delta n^2(i)}{2d}\log q_1 + \frac{2d - \delta n^2(i)}{2d}\log q_2 \\ &\quad - \frac{\chi(1 \leqslant \delta n^1(i) + \delta n^2(i) \leqslant d)}{2^{|\delta n^1(i) + \delta n^2(i)|}}\log q_1 - \frac{\chi(1 \leqslant \delta n^2(i) \leqslant d)}{2^{|\delta n^2(i)|}}\log q_2 \\ &\geqslant \frac{\log q_2}{2d} \end{aligned} \tag{A4}$$

so that each site $i \in \gamma$ such that $\delta n^2(i) \geqslant 1$ gives at least a contribution $(1/2d)\log q_2$ to the left-hand side of (23). Thus, each site of γ gives at least a contribution $(1/2d)\log q_0$ and we conclude the proof of the lemma.

ACKNOWLEDGMENTS

We wish to thank A. Messager for useful discussions. Part of this work was done when L.L. and N.M. visited the Centre de Physique Théorique-Marseille and when J.R. visited the Department of Mathematics of Rutgers University and the Courant Institute for Mathematical Sciences of New York University. The authors acknowledge the three organizations for warm hospitality and support. J.R. thanks the Mission des Relations Internationales-CNRS for financial support. This work was partially supported by agreement CNR Morocco–CNRS France.

REFERENCES

1. F. Dunlop, L. Laanait, A. Messager, S. Miracle-Sole, and J. Ruiz, Multilayer wetting in partially symmetric q-state models, *J. Stat. Phys.* **59**:1383–1396 (1991).
2. Ya. G. Sinai, *Theory of Phase Transitions: Rigorous Results* (Pergamon Press, London, 1982).
3. J. Bricmont, K. Kuroda, and J. L. Lebowitz, First order phase transitions in lattice and continuous systems, *Commun. Math. Phys.* **101**:501–538 (1985).
4. L. Laanait, A. Messager, S. Miracle-Sole, J. Ruiz, and S. Shlosman, Interfaces in the Potts model I: Pirogov–Sinai theory of the Fortuin–Kasteleyn representation, *Commun. Math. Phys.* **140**:81–91 (1991).
5. C. M. Fortuin and P. W. Kasteleyn, On the random cluster model, *Physica* **57**:536–564 (1972).
6. R. G. Edwards and A. D. Sokal, Generalization of the Fortuin–Kasteleyn–Swendsen–Wang representation and Monte Carlo algorithm, *Phys. Rev. B* **38**:2009–2013 (1988).
7. R. H. Swendsen and J. S. Wang, Nonuniversal critical dynamics and Monte Carlo simulations, *Phys. Rev. Lett.* **58**:86–89 (1987).
8. A. Benyoussef, L. Laanait, and M. Loulidi, More results on the Ashkin–Teller model, Preprint Marseille CPT-92/P.2655 (1992).
9. P. M. C. de Oliviera and F. C. Sa Barreto, Renormalization group studies of the Ashkin–Teller model, *J. Stat. Phys.* **57**:53–63 (1989).
10. C.-E. Pfister, Phase transitions in the Ashkin–Teller model, *J. Stat. Phys.* **29**:113–116 (1982).
11. R. Kotecký and D. Preiss, An inductive approach to Pirogov–Sinai theory, *Supp. Rend. Circ. Matem. Palermo* **II**(3):161–164 (1984).
12. N. Masaif, Étude du modèle $Z(q)$ partiellement symétrique, Diplôme de Troisième Cycle, Faculté des Sciences, Rabat, Morocco (1992).
13. M. Zahradník, An alternate version of Pirogov–Sinai theory, *Commun. Math. Phys.* **93**:559–581 (1984).
14. C. Borgs and J. Imbrie, A unified approach to phase diagram in field theory and statistical mechanics, *Commun. Math. Phys.* **123**:305–328 (1989).
15. C.-E. Pfister, Translation invariant equilibrium states of ferromagnetic Abelian lattice systems, *Commun. Math. Phys.* **86**:375–390 (1982).
16. C.-E. Pfister, On the ergodic decomposition of Gibbs random fields for ferromagnetic Abelian lattice models, *Ann. N. Y. Acad. Sci.* **491**:170–190 (1987).
17. M. Zahradník, Analitycity of low temperature phase diagrams of lattice spin models, *J. Stat. Phys.* **47**:725–755 (1987).

18. J. De Coninck, A. Messager, S. Miracle-Sole, and J. Ruiz, A study of perfect wetting for Potts and Blume–Capel models with correlation inequalities, *J. Stat. Phys.* **52**:45–60 (1988).
19. A. Messager, S. Miracle-Sole, J. Ruiz, and S. Shlosman, Interfaces in the Potts model II: Antonov's rule and rigidity of the order disorder interface, *Commun. Math. Phys.* **140**:275–290 (1991).
20. R. Kotecký, L. Laanait, A. Messager, and J. Ruiz, The q-state Potts model in the standard Pirogov–Sinai theory: Surface tensions and Wilson loops, *J. Stat. Phys.* **58**:199–248 (1990).